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FULLY PLASTIC CENTER-CRACKED STRIP UNDER ANTI-PLANE SHEAR†

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Abstract—The problem of a center-cracked strip subjected to uniform remote anti-plane shear stress is transformed to a problem in a hodograph plane which is solved exactly by Mellin transform and Wiener–Hopf technique. The material of the strip satisfies a pure power hardening stress strain relation and the results are valid for both deformation and flow theories of placticity. Numerical values are given for the crack opening displacement δ and Rice's path independent J integral for several values of the power hardening exponent n and crack width to strip width ratios. Approximate asymptotic formulas are presented for J and δ for large n.

1. INTRODUCTION

Recent experimental and analytic studies[1, 2] have demonstrated that Rice's J integral[3, 4] provides a good elastic-plastic fracture criterion. A number of studies (e.g.[5, 6]) have consequently been devoted to the accurate determination of J and the crack opening displacement. Bucci *et al.*[7], Rice *et al.*[8] and Shih[9] have proposed estimation procedures for the determination of J. The estimates in[9] are based on a solution by finite element technique of the same problem considered in this paper. Further discussions of the application of the results contained here are therefore unnecessary.

The edge-cracked strip subjected to remote anti-plane shear is analyzed by a hodograph transformation and Wiener–Hopf technique. Clearly the solution to the edge-cracked strip provides the solution to the center-cracked strip, double edge crack and periodically distributed collinear cracks. The method used here is presented more fully in[5] where the solution of the edge crack in a semi-infinite body is given.

2. DIFFERENTIAL EQUATION

We consider an infinite strip of isotropic solid of width b occupying the region $-a \le x \le b-a$ with an edge crack $y=0, -a\le x\le 0$ (see Fig. 1a). The body is subjected to out of plane uniform shear stress $\tau_{yz}=\tau_\infty$ at $y=\pm\infty$. The surfaces $x=-a$ and $x=b-a$ are stress free. For small strains the only nonvanishing components of displacement vector, strain and stress tensors are w, γ_{xz}, γ_{yz} and τ_{xz}, τ_{yz}, respectively. The strain-displacement relations are $\gamma_{xz}=\partial w/\partial x$, $\gamma_{yz}=\partial w/\partial y$. The compatibility and equilibrium equations are

$$\partial\gamma_{xz}/\partial y = \partial\gamma_{yz}/\partial x, \quad \text{and} \quad \partial\tau_{xz}/\partial x + \partial\tau_{yz}/\partial y = 0.$$

We assume a pure power hardening stress strain law

$$\gamma/\gamma_0 = \alpha(\tau/\tau_0)^n \tag{1}$$

where α is a nondimensional constant which can be omitted without any loss in generality. It is however retained for consistency with the literature. τ_0 and γ_0 are reference values of the principal stress $\tau=(\tau_{xz}^2+\tau_{yz}^2)^{1/2}$ and the principal strain $\gamma=(\gamma_{xx}^2+\gamma_{yz}^2)^{1/2}$. The resulting equation for w

$$\nabla^2 w = \frac{1}{2}\left(1-\frac{1}{n}\right)\nabla w \,.\, \nabla \ln(\nabla w \,.\, \nabla w) \tag{2}$$

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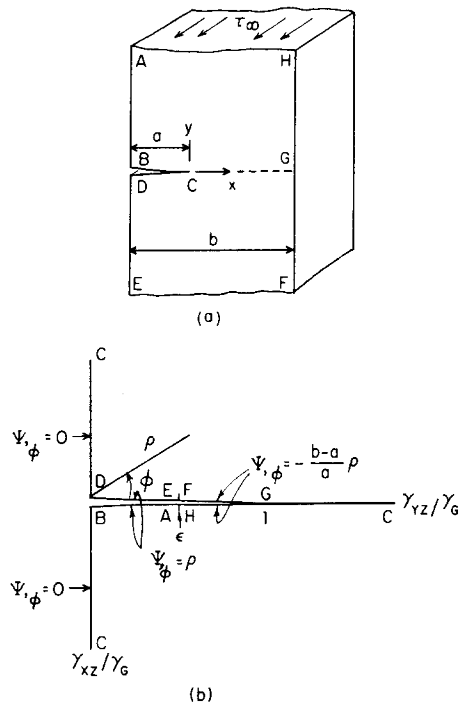

Fig. 1.

is nonlinear and homogeneous of degree 1 and is not readily solvable by analytic methods. It has been shown[10, 11, 5] that it is convenient to use a hodograph transformation (Fig. 1b). Following very closely the analysis in[5] we introduce a scalar potential function ψ such that

$$\mathbf{x} = \nabla_\gamma \psi \tag{3}$$

where $\mathbf{x}$ is the position vector and ∇_γ is the gradient operator with respect to the strain vector $\boldsymbol{\gamma} = (\gamma_{xz}, \gamma_{yz})$.

The differential equation and boundary conditions satisfied by ψ are

$$n\Psi_{,\rho\rho} + \frac{1}{\rho}\Psi_{,\rho} + \frac{1}{\rho^2}\Psi_{,\phi\phi} = 0 \qquad \rho > 0, \quad -\pi/2 < \phi < \pi/2 \tag{4}$$

$$\Psi_{,\phi}(\rho, 0^{\pm}) = \begin{cases} \rho & 0 < \rho < \epsilon \\ -c\rho & \epsilon < \rho < 1 \end{cases} \tag{5}$$

$$\Psi_{,\phi}(\rho, \pm\pi/2) = 0 \tag{6}$$

where

$$\Psi = \psi/a\gamma_G, \quad \epsilon = \gamma_\infty/\gamma_G < 1, \quad c = b/a - 1, \quad \Psi_{,\phi}(\rho, 0^+) \equiv \lim_{h\to 0} \Psi_\phi(\rho, h), \quad h > 0$$

$$\gamma_{xz}/\gamma_G = -\rho \sin\phi \tag{7}$$

$$\gamma_{yz}/\gamma_G = \rho \cos\phi$$

γ_∞ is the strain corresponding to the remotely applied stress and $\gamma_G = \gamma_{xz}$ at the point G: $(x, y) = (b - a, 0)$. A comma subscript denotes differentiation with respect to subsequent subscript(s). Note that ϵ depends on the solution hence the problem is nonlinear.

3. WIENER-HOPF PROBLEM

The problem consisting of eqns (4)–(6) can be solved by Mellin transform and Wiener–Hopf technique. Application of this technique requires knowledge of Ψ as $\rho \to \infty$ and $\rho \to 0$ given by[5]

$$\Psi \to \rho^{-1/n} \quad \text{as} \quad \rho \to \infty \tag{8}$$

and

$$\left.\begin{array}{l}\Psi_{,\phi} \to \rho \cos \phi \\ \Psi_{,\rho} \to \sin \phi\end{array}\right\} \quad \text{as} \quad \rho \to 0. \tag{9}$$

Now we introduce the Mellin transform $\bar{\Psi}$ of Ψ defined by

$$\bar{\Psi}(s, \phi) = \int_0^\infty \rho^{s-1} \Psi(\rho, \phi)\, \mathrm{d}\phi. \tag{10}$$

Taking the transform of (4)–(6) gives

$$\bar{\Psi}(s, \phi) = \left[\frac{\epsilon^{s+1}}{s+1} - c\frac{1-\epsilon^{s+1}}{s+1} + \bar{v}(s)\right] \frac{\cos \omega(s)(\phi - \pi/2)}{\omega(s) \sin \frac{\pi}{2} \omega(s)} \qquad 0 < Re\, s < 1/n, \quad \phi > 0 \tag{11}$$

where $\bar{v}(s)$ is the Mellin transform of

$$v(\rho) = \begin{cases} 0 & 0 < \rho < 1 \\ \Psi_{,\phi}(\rho, 0) & \rho > 1 \end{cases} \tag{12}$$

and

$$\omega^2(s) = s[n(s+1) - 1] \tag{13}$$

For

$$\phi < 0, \qquad \bar{\Psi}(s, -\phi) = -\bar{\Psi}(s, \phi) \tag{14}$$

Define

$$\bar{g}(s) = \bar{\Psi}(s, 0^+) - \bar{\Psi}(s, 0^-) \tag{15}$$

then from (11) and (14) we have

$$\frac{1}{2}\bar{g}(s) = \left[\frac{\epsilon^{s+1}}{s+1} - c\frac{1-\epsilon^{s+1}}{s+1} + \bar{v}(s)\right] p(s) \qquad 0 < Re\, s < 1/n \tag{16}$$

where

$$p(s) = \omega^{-1}(s) \cot \frac{\pi}{2} \omega(s). \tag{17}$$

We note that from (12) and (8) $\bar{v}(s)$ is analytic in the half-plane $Re\, s < 1/n$. Also since $\Psi(\rho, \phi)$ is continuous on the half-line $\phi = 0, \rho > 1$, then $g(\rho) = 0$ for $\rho > 1$ and from (9) $\bar{g}(s)$ is analytic for $Re\, s > 0$. Denoting functions analytic for $Re\, s > 0$ by subscript + and functions analytic for $Re\, s < 1/n$ by subscript − (16) becomes

$$\frac{1}{2}\bar{g}_+(s) = \left[\left(\frac{b}{a}\frac{\epsilon^{s+1}}{s+1} - \frac{c}{s+1}\right)_+ + \bar{v}_-(s)\right] p(s) \qquad 0 < Re\, s < 1/n. \tag{18}$$

It is noteworthy that the second term in the bracket in (11) is an entire function and thus in principle may be retained as an entity during the analysis. Nevertheless the application of Wiener–Hopf technique to the resulting equation gives an entire function with complicated behavior at $s = \infty$.

The solution of the Wiener–Hopf problem (18) is obtained in two steps. The first step consists of decomposing $p(s)$ into a quotient

$$p(s) = N_-(n, s)/D_+(n, s) \tag{19}$$

where $N_-(n, s)$ has no poles or zeros for $Re\, s < 1/n$, and $D_+(n, s)$ has no poles or zeros for $Re\, s > 0$. The functions N and D are given in [5] and we summarize their pertinent properties.

$$N_-(n, s) = 2^{-s\sqrt{(n)}} \prod_{k=0}^{\infty} \left(\gamma_k^+ - \frac{\sqrt{(n)}}{2k+1} s\right) \exp(\bar{s}\sqrt{(n)}/2k+1) \Big/ \prod_{k=1}^{\infty} (\beta_k^+ - s\sqrt{(n)}/2k) \exp(\bar{s}\sqrt{(n)}/2k) \tag{20}$$

$$D_+(n, s) = 2^{s\sqrt{(n)}-1} \pi s(s+1-1/n) \prod_{k=1}^{\infty} \left(\frac{\sqrt{(n)}}{2k} s - \beta_k^-\right) \exp(-\bar{s}\sqrt{(n)}/2k) \Big/ \prod_{k=0}^{\infty} \left(\frac{\sqrt{(n)}}{2k+1} s - \gamma_k^-\right) \exp(-\bar{s}\sqrt{(n)}/(2k+1)) \tag{21}$$

where

$$\bar{s} = s + (n-1)/2n$$

$$\gamma_k^{\pm} = n^{1/2}\{-1 + 1/n \pm [(1-1/n)^2 + 4(2k+1)^2/n]^{1/2}\}/2(2k+1) \tag{22}$$

$$\beta_k^{\pm} = n^{1/2}\{-1 + 1/n \pm [(1-1/n)^2 + 16k^2/n]^{1/2}\}/4k. \tag{23}$$

Asymptotically

$$N_-(n, s) \sim (-\pi s/2)^{1/2} n^{1/4} 2^{(n-1)/2\sqrt{(n)}} \quad \text{as} \quad |s| \to \infty, \quad 0 < \arg s < 2\pi \tag{24}$$

$$D_+(n, s) \sim (\pi/2)^{1/2} n^{3/4} s^{3/2} 2^{(n-1)/2\sqrt{(n)}} \quad \text{as} \quad |s| \to \infty, \quad -\pi < \arg s < \pi \tag{25}$$

Substituting (19) into (18) gives

$$\frac{1}{2}\bar{g}_+(s) D_+(n, s) = \left[\frac{b}{a}\frac{\epsilon^{s+1}}{s+1} - \frac{c}{s+1}\right] N_-(n, s) + \bar{v}_-(s) N_-(n, s) \tag{26}$$

Now the first set of terms on the right hand side of (26) must be decomposed into the sum of a + function and a − function. Noting that $N_-(n, s)$ has simple poles at $s = b_k^+$ where

$$b_k^{\pm} = 2k\beta_k^{\pm}/\sqrt{(n)} \tag{27}$$

a decomposition is accomplished by separating the principal parts of the Laurent series of $N_-(n, s)$ at these poles from the function. Thus

$$\frac{\epsilon^{s+1}}{s+1} N_-(n, s) = \left(\sum_{k=1}^{\infty} \frac{A_k}{s - b_k^+}\right)_- + \left(\frac{\epsilon^{s+1}}{s+1} N_-(n, s) - \sum_{k=1}^{\infty} \frac{A_k}{s - b_k^+}\right)_+ \tag{28}$$

where A_k is the residue of $\epsilon^{s+1} N_-(n, s)/(s+1)$ at $s = b_k^+$ and is given by

$$A_k = -\frac{2k\epsilon^{1+b_k^+} 2^{-\sqrt{(n)} b_k^+}}{\sqrt{(n)}(1+b_k^+) \exp(\sqrt{(n)}\bar{b}_k/2k)} \cdot \frac{\prod_{m=0}^{\infty} \left(\gamma_m^+ - \frac{\sqrt{(n)}}{2m+1} b_k^+\right) \exp(\sqrt{(n)}\bar{b}_k/(2m+1))}{\prod_{\substack{m=1 \\ m \neq k}}^{\infty} \left(\beta_m^+ - \frac{\sqrt{(n)}}{2m} b_k^+\right) \exp(\sqrt{(n)}\bar{b}_k/2m)} \tag{29}$$

where $\bar{b}_k = b_k^+ + \dfrac{n-1}{2n}$. The second term in the brackets in (26) is decomposed more readily as

$$\frac{N_-(n, s)}{s+1} = \left(\frac{N_-(n, -1)}{s+1}\right)_+ + \left(\frac{N_-(n, s) - N_-(n, -1)}{s+1}\right)_- \tag{30}$$

Although the decomposition used in (30) gives a simpler decomposition of the left hand side of (28), such a splitting complicates considerably the subsequent problem of determining the entire function.

Substituting the decompositions (28) and (30) into (26) gives

$$\frac{1}{2}\bar{g}_+(s)D_+(n,s)-\frac{b}{a}\left(\frac{\epsilon^{s+1}}{s+1}N_-(n,s)-\sum_{k=1}^{\infty}\frac{A_k}{s-b_k^+}\right)_+ +c\left(\frac{N_-(n,-1)}{s+1}\right)_+$$
$$=\frac{b}{a}\left(\sum_{k=1}^{\infty}\frac{A_k}{s-b_k^+}\right)_- -c\left(\frac{N_-(n,s)-N_-(n,-1)}{s+1}\right)_- +\bar{v}_-(s)N_-(n,s). \quad (31).$$

Now since the left hand side is analytic for $Re\, s>0$ and the right hand side is analytic for $Re\, s<1/n$ each must be the analytic continuation of the other. Hence each represents the same entire function $E(s)$. The entire function is determined by obtaining its asymptotic behavior as $|s|\to\infty$ in both half planes.

The dominant asymptotic behavior of $\bar{v}(s)$ and $\bar{g}_+(s)$ is determined by the behavior of $v(\rho)$ and $g(\rho)$ in the neighborhood of $\rho=1$. Introduce the variables r_1, β through the equations

$$\begin{aligned}-r_1\sin\beta &= n^{1/2}\gamma_{xz}/\gamma_G\\ r_1\cos\beta &= \gamma_{yz}/\gamma_G-1\end{aligned} \quad (32)$$

and seek the asymptotic behavior of Ψ as $\rho\to 1^{\pm}$ in the form

$$\Psi(r_1,\beta)=g_1(\beta)r_1^{1/2}+g_2(\beta)r_1+g_3(\beta)r_1^{3/2}+\dots. \quad (33)$$

Substituting (32), (33) and (7) in (4) and noting that by symmetry $\partial\gamma_y/\partial y=0$ (i.e. $\partial^2\psi/\partial\gamma_{xz}^2=0$) gives

$$g_1(\beta)\equiv 0,\quad g_2(\beta)=\frac{1}{\sqrt{(n)}}\sin\beta,\quad g_3(\beta)=\text{constant}\,.\,\sin\frac{3}{2}\alpha$$

Thus

$$\Psi_\phi(\rho,0)\sim\rho \quad\text{as}\quad \rho\to 1^+$$

and

$$\Psi(\rho,0^+)\sim(\text{constant})(1-\rho)^{3/2}\quad\text{as}\quad\rho\to 1^- \quad (34)$$

The use of (34) in the definition of $\bar{v}_-(s)$ and $\bar{g}_+(s)$ and application of Watson's lemma [12] gives

$$\begin{aligned}\bar{v}_-(s)&\sim-\frac{1}{s} &&\text{as}\quad |s|\to\infty,\quad Re\,s<1/n\\ \bar{g}_+(s)&\sim(\text{constant})\,.\,s^{-5/2} &&\text{as}\quad |s|\to\infty,\quad Re\,s>0\end{aligned} \quad (35)$$

From the asymptotic behaviors (24), (25) and (35) it follows that each side of (31) tends to zero as $|s|\to\infty$ in its domain of definition. Consequently by Liouville's theorem the entire function $E(s)$ is identically zero. Thus from (31)

$$\frac{b}{a}\frac{\epsilon^{s+1}}{s+1}-\frac{c}{s+1}+\bar{v}_-(s)=\frac{b}{a}\frac{\epsilon^{s+1}}{s+1}-\frac{1}{N_-(n,s)}\left[\frac{cN_-(n,-1)}{s+1}+\frac{b}{a}\sum_{k=1}^{\infty}\frac{A_k}{s-b_k^+}\right]$$

and substituting this result in (11) and taking the inverse Mellin transform gives

$$\Psi(\rho,\phi)=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\rho^{-s}\left\{\frac{b}{a}\frac{\epsilon^{s+1}}{s+1}-\frac{1}{N_-(n,s)}\left[\frac{cN_-(n,-1)}{s+1}+\frac{b}{a}\sum_{k=1}^{\infty}\frac{A_k}{s-b_k^+}\right]\right\}\frac{\cos\omega(s)(\phi-\pi/2)}{\omega(s)\sin\frac{\pi}{2}\omega(s)}\,ds$$

$$\phi>0,\quad 0<\nu<1/n. \quad (36)$$

We observe that the integrand has simple poles at $s = 1/n - 1, -1, 0, b_m^-, b_m^+, a_{m-1}^+, m = 1, 2, 3, \ldots$ where

$$a_m^{\pm} = (2m+1)\gamma^{\pm}/\sqrt{(n)}$$

In the neighborhood of $\rho = 0$ the pole $s = 1/n - 1$ gives a contribution to Ψ that contradicts (9). Thus we must eliminate this pole by a proper choice of ϵ. Setting the quantity in braces in (36) equal to zero for $s = 1/n - 1$ gives the following implicit relation for ϵ.

$$N_-\left(n, \frac{1}{n} - 1\right)\epsilon^{1/n} = (1 - a/b)N_-(n, -1) + \frac{1}{n}\sum_{k=1}^{\infty} \frac{A_k}{b_k^+ + 1 - 1/n}. \tag{37}$$

Note from (29) that A_k depends on ϵ.

It follows from the theory of residues that for $\phi > 0$

$$\Psi(\rho, \phi) = \begin{cases} \rho \sin\phi + \dfrac{2}{\pi(n-1)}\left\{\dfrac{b}{a}\epsilon - \left[cN_-(n,-1) - \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{b_k^+}\right]\dfrac{1}{N_-(n,0)}\right\} \\ \quad + \dfrac{1}{\pi}\displaystyle\sum_{m=1}^{\infty}\left[\dfrac{b}{a}\epsilon^{1+b_m^-} - \dfrac{cN_-(n,-1)}{1+b_m^-} - \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{b_k^+ - b_m^-}\right]\dfrac{\rho^{-b_m^-}}{m\omega'(b_m^-)N_-(n, b_m^-)}\cos 2m\phi & 0 < \rho < \epsilon \\ -c\rho\sin\phi - \dfrac{2}{\pi(n-1)}\left[cN_-(n,-1) - \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{b_k^+}\right]\dfrac{1}{N_-(n,0)} \\ \quad + \dfrac{2}{\pi(n-1)}\left[ncN_-(n,-1) - \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{b_k^+ + 1 - (1/n)}\right]\dfrac{\rho^{1-1/n}}{N_-(n, (1/n) - 1)} \\ \quad - \dfrac{1}{\pi}\displaystyle\sum_{m=1}^{\infty}\left[\dfrac{cN_-(n,-1)}{1+b_m^-} - \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{b_k^+ - b_m^-}\right]\dfrac{\rho^{-b_m^-}}{m\omega'(b_m^-)N_-(n, b_m^-)}\cos 2m\phi \\ \quad - \dfrac{\epsilon b}{\pi a}\displaystyle\sum_{m=1}^{\infty}\dfrac{1}{m(1+b_m^+)\omega'(b_m^+)}\left(\dfrac{\rho}{\epsilon}\right)^{-b_m^+}\cos 2m\phi & \epsilon < \rho < 1 \\ -\displaystyle\sum_{m=0}^{\infty}\dfrac{\left[c\dfrac{N_-(n,-1)}{1+a_m^+} + \dfrac{b}{a}\displaystyle\sum_{k=1}^{\infty}\dfrac{A_k}{a_m^+ - b_k^+}\right]2^{\sqrt{(n)}a_m^+}\displaystyle\prod_{p=1}^{\infty}\left(\beta_p^+ - \dfrac{\sqrt{(n)}}{2p}a_m^+\right)e^{[\sqrt{(n)}/2p]\bar{a}_m}}{\sqrt{(n)}\exp\left(\dfrac{\sqrt{(n)}}{2n+1}a_m^-\right)\displaystyle\prod_{\substack{p=0\\p\neq m}}^{\infty}\left(\gamma_p^+ - \dfrac{\sqrt{(n)}}{2p+1}a_m^+\right)e^{[\sqrt{(n)}/2p+1]\bar{a}_m}} \\ \quad \times \rho^{-a_m^+}\sin(2m+1)\phi & \rho > 1 \end{cases} \tag{38}$$

where $\bar{a}_m = a_m^+ + (n-1)/2n$, $\omega'(\cdot) = d\omega(\cdot)/ds$. Equations (37) and (38) give the exact solution to the problem in the hodograph plane.

4. J INTEGRAL AND CRACK OPENING DISPLACEMENT

The path independent integral, J, is defined by[3, 4]

$$J = \int_\Gamma \left[W dy - \mathbf{T} \cdot \frac{\partial \mathbf{u}}{\partial x} ds\right]$$

where Γ is any simple contour in the xy-plane encircling the crack tip, W is the strain energy density, $\mathbf{T}$ is the stress vector acting on Γ, and $\mathbf{u}$ is the displacement vector. In terms of the scalar potential function Ψ this integral becomes

$$J = -\frac{na\gamma_\infty^{1+1/n}\alpha^{-1/n}}{\gamma_0^{1/n}\epsilon^{1+1/n}} \cdot \rho^{2+1/n}\frac{\partial^2}{\partial\rho^2}\int_{-\pi/2}^{\pi/2}\Psi\sin\phi\, d\phi \qquad \rho > 1. \tag{39}$$

Substituting for Ψ from (38) and performing the trivial integration gives

$$J/a\tau_\infty\gamma_\infty = -\tfrac{1}{2}\pi(1 + 1/n)\epsilon^{-(1+1/n)}Q(n, \epsilon) \tag{40}$$

where

$$Q(n,\epsilon)=\frac{2^{1/\sqrt{n}}\left[\frac{b}{a}\sum_{k=1}^{\infty}\frac{A_k}{b_k^{+}-(1/n)}-\frac{ncN_{-}(n,-1)}{n+1}\right]\prod_{p=1}^{\infty}\left(\beta^{+}-\frac{1}{2p\sqrt{n}}\right)\exp\left((n+1)/4p\sqrt{n}\right)}{\sqrt{n}\exp\left((n+1)/2\sqrt{n}\right)\prod_{p=1}^{\infty}\left(\gamma_p^{+}-\frac{1}{(2p+1)\sqrt{n}}\right)\exp\left((n+1)/2(2p+1)\sqrt{n}\right)}. \tag{41}$$

(The expression (60) for $Q(n)$ in [5] should be divided by $\exp((n+1)/2\sqrt{(n)})$ and $2^{\sqrt{(n)}}$ should be $2^{1/\sqrt{(n)}}$).

The crack opening displacement δ is defined by

$$\delta = w(x=-a, y=0^{+}) - w(x=-a, y=0^{-})$$

and in terms of the scalar potential Ψ is

$$\delta = \frac{2a}{\epsilon}\gamma_\infty . \lim_{\rho\to 0}\left[\rho\Psi(\rho,0^{+})-\Psi(\rho,0^{+})\right]. \tag{42}$$

Substituting for Ψ from (38) gives

$$\frac{\delta}{a\gamma_\infty}=\frac{4b}{\pi a\epsilon(n-1)}\left\{\frac{1}{N_{-}(n,0)}\left[\left(1-\frac{a}{b}\right)N_{-}(n,-1)-\sum_{k=1}^{\infty}A_k/b_k^{+}\right]-\epsilon\right\}. \tag{43}$$

The following scheme proved useful in obtaining numerical values of J and δ for various values of the power hardening exponent n and crack width a to strip width b ratios. For each value of n and a/b eqn (37) is solved iteratively for ϵ to within 0·01% error. This value of ϵ is then used to compute J and δ. The product series were computed accurately by means of double precision arithmetic. The results of these computations for J and δ are given in Table 1 and plotted in Fig. 2.

Table 1.

(a) $\bar{J} = J(1-a/b)^n/(a\tau_\infty\gamma_\infty)$

a/b \ n	1	1.5	2	3	5	10	20	50
0*	1.571	1.939	2.271	2.864	3.865	5.788	8.524	13.87
1/8	1.392	1.636	1.791	2.023	2.245	2.265	1.936	1.325
1/4	1.243	1.369	1.444	1.504	1.482	1.282	.995	.663
1/2	1.000	1.003	.982	.924	.817	.652	.498	.331
3/4	.805	.750	.703	.634	.547	.435	.332	.221
7/8	.718	.653	.607	.544	.469	.373	.284	.189
1	.637	.573	.531	.476	.411	.326	.249	.166

(b) $\bar{\delta} = (1-a/b)^n/a\gamma_\infty$

a/b \ n	1	1.5	2	3	5	10	20	50
0*	2.000	2.334	2.644	3.209	4.175	6.045	8.727	14.01
1/8	1.761	1.944	2.063	2.234	2.373	2.273	1.852	1.206
1/4	1.571	1.592	1.612	1.591	1.464	1.152	.824	.517
1/2	1.122	1.025	.934	.784	.594	.398	.275	.172
3/4	.685	.529	.424	.304	.205	.133	.0917	.0575
7/8	.422	.280	.205	.135	.0880	.0570	.0393	.0246

*Results for a/b = 0 are taken from [5], where n = 30, 50, 100 should be n = 50, 100, 1000.

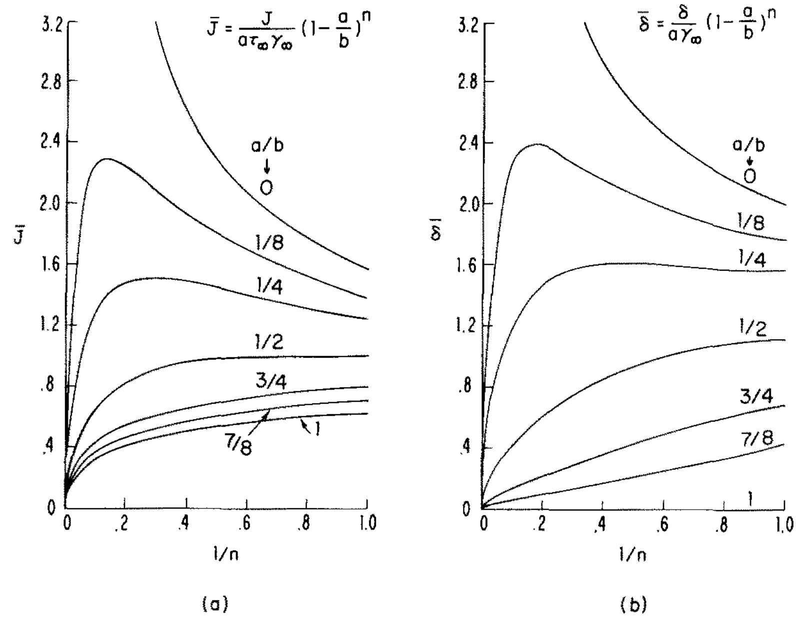

Fig. 2.

As shown in[6] convenient normalizations for J and δ are

$$\bar{J} = \frac{J}{a\tau_\infty\gamma_\infty}\left(1-\frac{a}{b}\right)^n = \frac{J/(\tau_0\gamma_0 a)}{\alpha\left(\frac{b-a}{b}\right)\left(\frac{P}{P_{\text{Limit}}}\right)^{n+1}} \tag{44}$$

and

$$\dot{\delta} = \frac{\delta}{a\gamma_\infty}\left(1-\frac{a}{b}\right)^n = \frac{\delta/\gamma_0 a)}{\alpha\left(\frac{P}{P_{\text{Limit}}}\right)^n}$$

where $P = \tau_\infty b$ is the total shear load per unit thickness of the strip and $P_{\text{Limit}} = \tau_0(b-a)$ is the limit load for a perfectly plastic strip ($n = \infty$).

For $a/b \to 1$ (37) has the asymptotic solution

$$\epsilon^{1/n} = (1-a/b)N_-(n,-1)/N_-(n,-1+1/n) + 0((1-a/b))$$

With this value for ϵ the exact expression for $\bar{J}$ was computed and the results are given in Table 1a and graphed in Fig. 2a. $\bar{\delta} \to 0$ as $a/b \to 1$.

For $n \gg 1$ the behavior of $\bar{\delta}$ and $\bar{J}$ are approximately given by the following unproved formula

$$\bar{\delta} \approx \left(\frac{\pi}{2}\right)^{3/2} \frac{b/a - 1}{\sqrt{((n-1)e)}} \tag{45}$$

and

$$\bar{J} \approx \left(\frac{\pi}{2}\right)^{3/2} \frac{b/a}{\sqrt{(n)}\exp((n+1)/2n)}. \tag{46}$$

Equations (45) and (46) which are not uniformly valid as $a/b \to 0$ give results that are in error by less than 5% for values of n and a/b to the right of the darkened grid in Table 1. These simple formulas may prove important in estimation procedures.

For the elastic strip ($n = 1$) the product expansions and series were also evaluated analytically giving known results[4, 13]

$$J/(a\tau_\infty\gamma_\infty) = (b/a)\tan(\pi a/2b) \tag{47}$$

and

$$\delta/(a\gamma_\infty) = (2b/\pi a)\ ln\ [(1+\sin(\pi a/2b))/(1-\sin(\pi a/2b))].$$

A comparison of the exact results in Table 1 with those of[9] reveals that the finite element technique used by Shih is very accurate. The maximum error ranges from 0·6% for $n = 2$ to 6% for $n = 10$. The near tip behavior given in[5] in terms of J for the center-cracked infinite solid remain valid for the strip except that the appropriate J must be used.

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